

Criteria of Biholomorphic Convex Mappings on the bounded convex balanced domain D_p^n

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Abstract. In this paper, we first establish several general sufficient conditions for the biholomorphic convex mappings on the bounded convex balanced domain D_p^n ($p_j \geq 2, j = 1, \dots, n$) in C^n , which extend some related results of earlier authors. From these, some concrete examples of biholomorphic convex mappings on D_p^n are also provided.

Keywords: Locally biholomorphic mapping, biholomorphic convex mapping, bounded convex balanced domain

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1 Introduction and Preliminaries

Suppose that C^n is the vector space of n complex variables $z = (z_1, z_2, \dots, z_n)$ with the Euclidean inner product $\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$, where $w = (w_1, w_2, \dots, w_n) \in C^n$. If Ω is a domain in C^n , and for every $z \in \Omega$, $|\lambda| \leq 1$, we have $\lambda z \in \Omega$, then we call Ω a balanced domain. Its Minkowski functional is defined by

$$\rho(z) = \inf\{t > 0, \frac{z}{t} \in \Omega\}, \quad z \in C^n.$$

Assume Ω is a bounded convex balanced domain in C^n , let $\rho(z)$ be its Minkowski functional, then $\rho(\cdot)$ is a norm of C^n , $\rho(\lambda z) = |\lambda|\rho(z)$ for every $\lambda \in C, z \in C^n$, and

$$\Omega = \{z \in C^n : \rho(z) < 1\},$$

and $\rho(z) = 0$ if and only if $z = 0$.

Let $p = (p_1, \dots, p_n)$ with $p_j > 1$ ($j = 1, 2, \dots, n$), and

$$D_p^n = \{(z_1, z_2, \dots, z_n) \in C^n : \sum_{j=1}^n |z_j|^{p_j} < 1\}, \quad (1)$$

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then D_p^n is a bounded convex balanced domain in C^n , and its Minkowski functional $\rho(z)$ satisfies the following equality

$$\sum_{j=1}^n \left| \frac{z_j}{\rho(z)} \right|^{p_j} = 1. \quad (2)$$

When $p_1 = \cdots = p_n = p$, we denote D_p^n by B_p^n , at this time, we have $\rho(z) = \sqrt[p]{|z_1|^p + \cdots + |z_n|^p}$. In particular, let $U = B_p^1$ be the unit disk in the complex plane C .

Let Ω be a domain in C^n , a mapping $f : \Omega \rightarrow C^n$ is said to be locally biholomorphic in Ω if f has a locally inverse at each point $z \in \Omega$ or, equivalently, if the first Fréchet derivative

$$Df(z) = \left(\frac{\partial f_j(z)}{\partial z_k} \right)_{1 \leq j, k \leq n}$$

is nonsingular at each point in Ω . The second Fréchet derivative of a mapping $f : \Omega \rightarrow C^n$ is a symmetric bilinear operator $D^2f(z)(\cdot, \cdot)$ on $C^n \times C^n$, and $D^2f(z)(z, \cdot)$ is the linear operator obtained by restricting $D^2f(z)$ to $\{z\} \times C^n$. The matrix representation of $D^2f(z)(b, \cdot)$ is

$$D^2f(z)(b, \cdot) = \left(\sum_{l=1}^n \frac{\partial^2 f_j(z)}{\partial z_k \partial z_l} b_l \right)_{1 \leq j, k \leq n}$$

where $f(z) = (f_1(z), \dots, f_n(z))$, $b = (b_1, \dots, b_n) \in C^n$.

Let $N(D_p^n)$ be the class of all locally biholomorphic mappings $f(z) = (f_1(z), \dots, f_n(z)) : D_p^n \rightarrow C^n$ such that $f(0) = 0$, $Df(0) = I$, where $z = (z_1, \dots, z_n) \in C^n$ and I is the unit matrix of $n \times n$. If $f \in N(D_p^n)$ is a biholomorphic mapping on D_p^n and $f(D_p^n)$ is a convex domain in C^n , then we say that f is a biholomorphic convex mappings on D_p^n . Let $K(D_p^n)$ denote the class of all biholomorphic convex mappings on D_p^n with $f(0) = 0$, $Df(0) = I$.

It is not easy to construct concrete biholomorphic convex mappings on some domains in C^n , even on the unit ball B_2^n . In 1995, Roper and Suffridge[7] proved that: If $f \in K$ and $F(z) = (f(z_1), \sqrt{f'(z_1)}z_0)$, where $z = (z_1, z_0) \in B_2^n$, $z_1 \in U$, $z_0 = (z_2, \dots, z_n) \in C^{n-1}$, then $F \in K(B_2^n)$. Which is popularly referred to as the Roper-Suffridge operator. Using this operator, we may construct a lot of concrete biholomorphic convex mappings on B_2^n . Gong and Liu [2] generalized the Roper-Suffridge operator to the Reinhardt domain $D_p = \{z = (z_1, z_2, \dots, z_n) \in C^n : \sum_{j=1}^n |z_j|^{p_j} < 1\}$, where $p_1 = 2, p_2 \geq 1, p_3 \geq 1, \dots, p_n \geq 1$. However, according to the result in [1, 9], none of these concrete examples belongs to $K(D_p^n)$ ($p_j > 2, j = 1, 2, \dots, n$). In 1999, Roper and Suffridge[8] gave two concrete examples of biholomorphic convex mappings on B_p^2 in C^n , Liu and Zhu[4] gave some sufficient conditions for biholomorphic convex mappings on B_p^n . Hamada and Kohr[3], Zhu[10] gave a necessary and sufficient condition for biholomorphic convex mappings on bounded convex balanced domain D_p^n as follows.

Theorem A[3, 10] Suppose that $p_j \geq 2 (j = 1, 2, \dots, n)$, $\rho(z)$ is the Minkowski functional of D_p^n , and $f \in N(D_p^n)$. Then $f \in K(D_p^n)$ if and only if for any $z = (z_1, z_2, \dots, z_n) \in$

$D_p^n \setminus \{0\}$, and $b = (b_1, b_2, \dots, b_n) \in C^n \setminus \{0\}$ such that

$$\operatorname{Re} \left\{ \sum_{j=1}^n p_j \left| \frac{z_j}{\rho(z)} \right|^{p_j} \frac{b_j}{z_j} \right\} = 0,$$

we have

$$\begin{aligned} J_f(z, b) &= \operatorname{Re} \left\{ \sum_{j=1}^n \frac{p_j^2}{2} \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 + \sum_{j=1}^n p_j \left(\frac{p_j}{2} - 1 \right) \left| \frac{z_j}{\rho(z)} \right|^{p_j} \left(\frac{b_j}{z_j} \right)^2 \right. \\ &\quad \left. - 2 \sum_{j=1}^n \frac{p_j}{\rho(z)} \left| \frac{z_j}{\rho(z)} \right|^{p_j} \left\langle Df(z)^{-1} D^2 f(z)(b, b), \frac{\partial \rho}{\partial \bar{z}} \right\rangle \right\} \geq 0. \end{aligned}$$

Liu and Zhu[5] had established some sufficient conditions of biholomorphic convex mappings on D_p^n for mappings of the following forms

$$f(z) = (f_1(z_1) + g_1(z_n), f_2(z_2) + g_2(z_n), \dots, f_{n-1}(z_{n-1}) + g_{n-1}(z_n), f_n(z_n)),$$

and

$$f(z) = (f_1(z_1, z_2, \dots, z_n), f_2(z_2), \dots, f_n(z_n)).$$

Liu and Li [6] did the further promotion on D_p^n , they had established a sufficient condition of biholomorphic convex mappings on D_p^n for mappings of the following form

$$f(z) = (f_1(z_1, z_2, \dots, z_n), f_2(z_2) + g_2(z_n), \dots, f_{n-1}(z_{n-1}) + g_{n-1}(z_n), f_n(z_n)).$$

Now we pose two problems as follows:

Problem I: Can we get some sufficient conditions such that mapping of the form :

$$f(z) = (f_1(z_1, z_2, \dots, z_n), f_2(z_2, \dots, z_n), \dots, f_{n-1}(z_{n-1}, z_n), f_n(z_n))$$

is a biholomorphic convex mapping on D_p^n ?

Problem II: Can we get some sufficient conditions such that the mapping

$$f(z) = (f_1(z_1, z_2), f_2(z_1, z_2)), \quad z = (z_1, z_2)$$

is a biholomorphic convex mapping on D_p^2 in C^2 ?

The object of this paper is to give partial answers to the above problems. From these, we may construct some concrete biholomorphic convex mappings on D_p^n .

2 Main results

In order to give a partial answer to Problem I, we first establish the following theorem.

Theorem 1 Suppose that $n \geq 2, p_j \geq 2, j = 1, 2, \dots, n$, and $f_j : U \rightarrow C$ is holomorphic with $f_j(0) = 0, f'_j(0) = 1 (j = 3, \dots, n-1, n)$, $f_1(z_1, \dots, z_n) : D_p^n \rightarrow C$ is holomorphic with $f_1(0, 0, \dots, 0) = 0, \frac{\partial f_1}{\partial z_1}(0, 0, \dots, 0) = 1, \frac{\partial f_1}{\partial z_l}(0, 0, \dots, 0) = 0 (l = 2, 3, \dots, n)$. Let

$$f(z) = (f_1(z_1, z_2, \dots, z_n), f_2(z_2, z_3, \dots, z_n), f_3(z_3), \dots, f_{n-1}(z_{n-1}), f_n(z_n)).$$

If $f(z)$ satisfies the following conditions

$$\begin{aligned}
(1) \quad & \frac{\partial f_1}{\partial z_1} \frac{\partial f_2}{\partial z_2} \prod_{j=3}^n f'_j(z_j) \neq 0, \quad |z_j f''_j(z_j)| \leq |f'_j(z_j)| (j = 3, 4, \dots, n); \\
(2) \quad & \sum_{l=1}^n |z_1 \frac{\partial^2 f_1}{\partial z_1 \partial z_l}| \leq |\frac{\partial f_1}{\partial z_1}|, \quad \sum_{l=2}^n |z_2 \frac{\partial^2 f_2}{\partial z_2 \partial z_l}| \leq |\frac{\partial f_2}{\partial z_2}|; \\
(3) \quad & p_1 \left(\sum_{l=1}^n \left| \frac{\frac{\partial^2 f_1}{\partial z_2 \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| + \sum_{l=2}^n \left| \frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_2}{\partial z_2}} \right| \left| \frac{\frac{\partial^2 f_2}{\partial z_2 \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| \right) \leq \left(1 - \sum_{l=2}^n \left| \frac{z_2 \frac{\partial^2 f_2}{\partial z_2 \partial z_l}}{\frac{\partial f_2}{\partial z_2}} \right| \right) p_2 |z_2|^{p_2-2}; \\
(4) \quad & p_1 \left(\sum_{l=1}^n \left| \frac{\frac{\partial^2 f_1}{\partial z_j \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| + \sum_{l=2}^n \left| \frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_2}{\partial z_2}} \right| \left| \frac{\frac{\partial^2 f_2}{\partial z_j \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| + \left| \frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_1}{\partial z_1}} \right| \left| \frac{\frac{\partial f_2}{\partial z_j}}{\frac{\partial f_2}{\partial z_2}} \right| \left| \frac{f''_j(z_j)}{f'_j(z_j)} \right| + \left| \frac{\frac{\partial f_1}{\partial z_j}}{\frac{\partial f_1}{\partial z_1}} \right| \left| \frac{f''_j(z_j)}{f'_j(z_j)} \right| \right) \\
& + p_2 \left(\sum_{l=2}^n \left| \frac{\frac{\partial^2 f_2}{\partial z_j \partial z_l}}{\frac{\partial f_2}{\partial z_2}} \right| + \left| \frac{\frac{\partial f_2}{\partial z_j}}{\frac{\partial f_2}{\partial z_2}} \right| \left| \frac{f''_j(z_j)}{f'_j(z_j)} \right| \right) \leq p_j |z_j|^{p_j-2} \left(1 - \left| z_j \frac{f''_j(z_j)}{f'_j(z_j)} \right| \right) (j = 3, 4, \dots, n)
\end{aligned}$$

for all $z = (z_1, \dots, z_n) \in D_p^n \setminus \{0\}$, then $f \in K(D_p^n)$.

Proof By direct computation of the Fréchet derivatives of $f(z)$, we get that

$$\begin{aligned}
Df(z) &= \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \dots & \frac{\partial f_1}{\partial z_{n-1}} & \frac{\partial f_1}{\partial z_n} \\ 0 & \frac{\partial f_2}{\partial z_2} & \dots & \frac{\partial f_2}{\partial z_{n-1}} & \frac{\partial f_2}{\partial z_n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & f'_{n-1}(z_{n-1}) & 0 \\ 0 & 0 & \dots & 0 & f'_n(z_n) \end{pmatrix} \\
Df(z)^{-1} &= \begin{pmatrix} \frac{1}{\frac{\partial f_1}{\partial z_1}} & -\frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_1}{\partial z_1} \frac{\partial f_2}{\partial z_2}} & \frac{\frac{\partial f_1}{\partial z_2} \frac{\partial f_2}{\partial z_3} - \frac{\partial f_1}{\partial z_3} \frac{\partial f_2}{\partial z_2}}{\frac{\partial f_1}{\partial z_1} \frac{\partial f_2}{\partial z_2} f'_3(z_3)} & \dots & \frac{\frac{\partial f_1}{\partial z_2} \frac{\partial f_2}{\partial z_{n-1}} - \frac{\partial f_1}{\partial z_{n-1}} \frac{\partial f_2}{\partial z_2}}{\frac{\partial f_1}{\partial z_1} \frac{\partial f_2}{\partial z_2} f'_{n-1}(z_{n-1})} & \frac{\frac{\partial f_1}{\partial z_2} \frac{\partial f_2}{\partial z_n} - \frac{\partial f_1}{\partial z_n} \frac{\partial f_2}{\partial z_2}}{\frac{\partial f_1}{\partial z_1} \frac{\partial f_2}{\partial z_2} f'_n(z_n)} \\ 0 & \frac{1}{\frac{\partial f_2}{\partial z_2}} & -\frac{\frac{\partial f_2}{\partial z_3}}{\frac{\partial f_2}{\partial z_2} f'_3(z_3)} & \dots & -\frac{\frac{\partial f_2}{\partial z_{n-1}}}{\frac{\partial f_2}{\partial z_2} f'_{n-1}(z_{n-1})} & -\frac{\frac{\partial f_2}{\partial z_n}}{\frac{\partial f_2}{\partial z_2} f'_n(z_n)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \frac{1}{f'_{n-1}(z_{n-1})} & 0 \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{f'_n(z_n)} \end{pmatrix} \\
D^2 f(z)(b, b) &= \begin{pmatrix} \sum_{l=1}^n \frac{\partial^2 f_1}{\partial z_1 \partial z_l} b_l & \sum_{l=1}^n \frac{\partial^2 f_1}{\partial z_2 \partial z_l} b_l & \dots & \sum_{l=1}^n \frac{\partial^2 f_1}{\partial z_n \partial z_l} b_l \\ 0 & \sum_{l=2}^n \frac{\partial^2 f_2}{\partial z_2 \partial z_l} b_l & \dots & \sum_{l=2}^n \frac{\partial^2 f_2}{\partial z_n \partial z_l} b_l \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & f''_n(z_n) b_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n \sum_{l=1}^n \frac{\partial^2 f_1}{\partial z_j \partial z_l} b_l b_j \\ \sum_{j=2}^n \sum_{l=2}^n \frac{\partial^2 f_2}{\partial z_j \partial z_l} b_l b_j \\ f''_3(z_3) b_3^2 \\ \dots \\ f''_n(z_n) b_n^2 \end{pmatrix}.
\end{aligned}$$

From (2) and $|z_j|^{p_j} = z_j^{\frac{p_j}{2}} \bar{z}_j^{\frac{p_j}{2}}$, direct computation yields

$$\frac{\partial \rho}{\partial \bar{z}_l} = \frac{p_l |z_l|^{p_l}}{2 \bar{z}_l \rho(z)^{p_l-1} \sum_{j=1}^n p_j \left| \frac{z_j}{\rho(z)} \right|^{p_j}}, \quad (3)$$

and $\left| \frac{z_j}{\rho(z)} \right| \leq 1 (j = 1, 2, \dots, n)$ for all $z = (z_1, z_2, \dots, z_n) \in D_p^n$.

Taking $z = (z_1, \dots, z_n) \in D_p^n \setminus \{0\}, b = (b_1, \dots, b_n) \in C^n \setminus \{0\}$ such that

$$\operatorname{Re} \left\{ \sum_{j=1}^n p_j \left| \frac{z_j}{\rho(z)} \right|^{p_j} \frac{b_j}{z_j} \right\} = 0,$$

by applying the fact $0 < \rho(z) < 1$ and the hypothesis of Theorem 1, we have

$$\begin{aligned} J_f(z, b) &\geq \sum_{j=1}^n \frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - 2 \sum_{j=1}^n \frac{p_j}{\rho(z)} \left| \frac{z_j}{\rho(z)} \right|^{p_j} \operatorname{Re} \langle Df(z)^{-1} D^2 f(z)(b, b), \frac{\partial \rho}{\partial \bar{z}} \rangle \\ &= \sum_{j=1}^n \frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - \operatorname{Re} \left\{ \frac{1}{\frac{\partial f_1}{\partial z_1}} \left[\sum_{j=1}^n \sum_{l=1}^n \frac{\partial^2 f_1}{\partial z_j \partial z_l} b_l b_j - \sum_{j=2}^n \sum_{l=2}^n \frac{\partial f_1}{\partial z_2} \frac{\partial^2 f_2}{\partial z_j \partial z_l} b_j b_l \right. \right. \\ &\quad \left. \left. + \sum_{j=3}^n \left(\frac{\partial f_1}{\partial z_2} \frac{\partial f_2}{\partial z_j} - \frac{\partial f_1}{\partial z_j} \right) \frac{f_j''(z_j)}{f_j'(z_j)} b_j^2 \right] \frac{p_1 |z_1|^{p_1}}{z_1 \rho(z)^{p_1}} \right. \\ &\quad \left. + \left[\sum_{j=2}^n \sum_{l=2}^n \frac{\partial^2 f_2}{\partial z_j \partial z_l} b_l b_j - \sum_{j=3}^n \frac{\partial f_2}{\partial z_2} \frac{f_j''(z_j)}{f_j'(z_j)} b_j^2 \right] \frac{p_2 |z_2|^{p_2}}{z_2 \rho(z)^{p_2}} + \sum_{j=3}^n \frac{f_j''(z_j)}{f_j'(z_j)} b_j^2 \frac{p_j |z_j|^{p_j}}{z_j \rho(z)^{p_j}} \right\} \\ &\geq \sum_{j=1}^n \frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - \frac{1}{\left| \frac{\partial f_1}{\partial z_1} \right|} \left[\sum_{j=1}^n \sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_j \partial z_l} \right| |b_j|^2 + \sum_{j=2}^n \sum_{l=2}^n \left| \frac{\partial f_1}{\partial z_2} \frac{\partial^2 f_2}{\partial z_j \partial z_l} \right| |b_j|^2 \right. \\ &\quad \left. + \sum_{j=3}^n \left(\frac{\left| \frac{\partial f_1}{\partial z_2} \right| \left| \frac{\partial f_2}{\partial z_j} \right|}{\left| \frac{\partial f_2}{\partial z_2} \right|} + \left| \frac{\partial f_1}{\partial z_j} \right| \right) \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| |b_j|^2 \right] \frac{p_1 |z_1|^{p_1-1}}{\rho(z)^{p_1}} - \sum_{j=3}^n \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| |b_j|^2 \frac{p_j |z_j|^{p_j-1}}{\rho(z)^{p_j}} \\ &\quad - \left[\sum_{j=2}^n \sum_{l=2}^n \left| \frac{\partial^2 f_2}{\partial z_j \partial z_l} \right| |b_j|^2 + \sum_{j=3}^n \left| \frac{\partial f_2}{\partial z_2} \frac{f_j''(z_j)}{f_j'(z_j)} \right| |b_j|^2 \right] \frac{p_2 |z_2|^{p_2-1}}{\rho(z)^{p_2}} \\ &= |b_1|^2 \frac{p_1 |z_1|^{p_1-2}}{\rho(z)^{p_1}} \left(1 - \sum_{l=1}^n \left| \frac{z_1 \partial^2 f_1}{\partial z_1 \partial z_l} \right| \right) + \frac{|b_2|^2}{\rho(z)} \left[\left(1 - \sum_{l=2}^n \left| \frac{z_2 \partial^2 f_2}{\partial z_2 \partial z_l} \right| \right) \frac{p_2 |z_2|^{p_2-2}}{\rho(z)^{p_2-1}} \right. \\ &\quad \left. - p_1 \left(\frac{|z_1|}{\rho(z)} \right)^{p_1-1} \left(\sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_2 \partial z_l} \right| + \sum_{l=2}^n \left| \frac{\partial f_1}{\partial z_2} \right| \left| \frac{\partial^2 f_2}{\partial z_2 \partial z_l} \right| \right) \right] + \sum_{j=3}^n \frac{|b_j|^2}{\rho(z)} \left[\frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j-1}} \left(1 - \left| \frac{z_j f_j''(z_j)}{f_j'(z_j)} \right| \right) \right. \\ &\quad \left. - p_1 \left(\frac{|z_1|}{\rho(z)} \right)^{p_1-1} \left(\sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_j \partial z_l} \right| + \sum_{l=2}^n \left| \frac{\partial f_1}{\partial z_2} \right| \left| \frac{\partial^2 f_2}{\partial z_j \partial z_l} \right| + \left| \frac{\partial f_1}{\partial z_1} \right| \left| \frac{\partial f_2}{\partial z_2} \right| \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| + \left| \frac{\partial f_1}{\partial z_1} \right| \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| \right) \right. \\ &\quad \left. - p_2 \left(\frac{|z_2|}{\rho(z)} \right)^{p_2-1} \left(\sum_{l=2}^n \left| \frac{\partial^2 f_2}{\partial z_j \partial z_l} \right| + \left| \frac{\partial f_2}{\partial z_2} \right| \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| \right) \right] \\ &\geq |b_1|^2 \frac{p_1 |z_1|^{p_1-2}}{\rho(z)^{p_1}} \left(1 - \sum_{l=1}^n \left| \frac{z_1 \partial^2 f_1}{\partial z_1 \partial z_l} \right| \right) + |b_2|^2 \left[p_2 |z_2|^{p_2-2} \left(1 - \sum_{l=2}^n \left| \frac{z_2 \partial^2 f_2}{\partial z_2 \partial z_l} \right| \right) \right. \\ &\quad \left. - p_1 \left(\sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_2 \partial z_l} \right| + \sum_{l=2}^n \left| \frac{\partial f_1}{\partial z_2} \right| \left| \frac{\partial^2 f_2}{\partial z_2 \partial z_l} \right| \right) \right] + \sum_{j=3}^n |b_j|^2 \left[p_j |z_j|^{p_j-2} \left(1 - \left| \frac{z_j f_j''(z_j)}{f_j'(z_j)} \right| \right) \right] \end{aligned}$$

$$\begin{aligned}
& -p_1 \left(\sum_{l=1}^n \left| \frac{\frac{\partial^2 f_1}{\partial z_j \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| + \sum_{l=2}^n \left| \frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_2}{\partial z_2}} \left| \frac{\frac{\partial^2 f_2}{\partial z_j \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| + \left| \frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_2}{\partial z_2}} \left| \frac{\frac{\partial f_2}{\partial z_j}}{f_j'(z_j)} \right| + \left| \frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_2}{\partial z_2}} \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| \right) \right. \\
& \left. - p_2 \left(\sum_{l=2}^n \left| \frac{\frac{\partial^2 f_2}{\partial z_j \partial z_l}}{\frac{\partial f_2}{\partial z_2}} \right| + \left| \frac{\frac{\partial f_2}{\partial z_j}}{\frac{\partial f_2}{\partial z_2}} \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| \right) \right) \geq 0.
\end{aligned}$$

Thus it follows from Theorem A that $f \in K(D_p^n)$. \square

Remark 1 Setting $f_2(z_2, \dots, z_n) = f_2(z_2)$ in Theorem 1, we get Theorem 2 of [5].

Setting $n = 3$ in Theorem 1, we get the following corollary, which gives an answer to Problem I for the case $n = 3$.

Corollary 1 Suppose that $p_j \geq 2, j = 1, 2, 3$. Let $f_3 : U \rightarrow C$ is holomorphic with $f_3(0) = 0, f_3'(0) = 1, f_1(z_1, z_2, z_3), f_2(z_2, z_3) : D_p^3 \rightarrow C$ are holomorphic with $f_1(0, 0, 0) = 0, \frac{\partial f_1}{\partial z_1}(0, 0, 0) = 1, \frac{\partial f_1}{\partial z_l}(0, 0, 0) = 0 (l = 2, 3)$ and $f_2(0, 0) = 0, \frac{\partial f_2}{\partial z_2}(0, 0) = 1, \frac{\partial f_2}{\partial z_3}(0, 0) = 0$. Set

$$f(z) = (f_1(z_1, z_2, z_3), f_2(z_2, z_3), f_3(z_3)).$$

If $f(z)$ satisfies the following conditions

- (1) $\frac{\partial f_1}{\partial z_1} \frac{\partial f_2}{\partial z_2} f_3'(z_3) \neq 0, \quad |z_3 f_3''(z_3)| \leq |f_3'(z_3)|;$
- (2) $\sum_{l=1}^3 \left| z_1 \frac{\partial^2 f_1}{\partial z_1 \partial z_l} \right| \leq \left| \frac{\partial f_1}{\partial z_1} \right|, \quad \sum_{l=2}^3 \left| z_2 \frac{\partial^2 f_2}{\partial z_2 \partial z_l} \right| \leq \left| \frac{\partial f_2}{\partial z_2} \right|;$
- (3) $p_1 \left(\sum_{l=1}^3 \left| \frac{\frac{\partial^2 f_1}{\partial z_2 \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| + \sum_{l=2}^3 \left| \frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_2}{\partial z_2}} \left| \frac{\frac{\partial^2 f_2}{\partial z_2 \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| \right) \leq \left(1 - \sum_{l=2}^3 \left| \frac{z_2 \frac{\partial^2 f_2}{\partial z_2 \partial z_l}}{\frac{\partial f_2}{\partial z_2}} \right| \right) p_2 |z_2|^{p_2-2};$
- (4) $p_1 \left(\sum_{l=1}^3 \left| \frac{\frac{\partial^2 f_1}{\partial z_3 \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| + \sum_{l=2}^3 \left| \frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_2}{\partial z_2}} \left| \frac{\frac{\partial^2 f_2}{\partial z_3 \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| + \left| \frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_2}{\partial z_2}} \left| \frac{f_3''(z_3)}{f_3'(z_3)} \right| + \left| \frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_2}{\partial z_2}} \left| \frac{f_j''(z_3)}{f_j'(z_3)} \right| \right) \right. \right. \\ \left. \left. + p_2 \left(\sum_{l=2}^n \left| \frac{\frac{\partial^2 f_2}{\partial z_3 \partial z_l}}{\frac{\partial f_2}{\partial z_2}} \right| + \left| \frac{\frac{\partial f_2}{\partial z_3}}{\frac{\partial f_2}{\partial z_2}} \left| \frac{f_3''(z_3)}{f_3'(z_3)} \right| \right) \right) \leq p_3 |z_3|^{p_3-2} \left(1 - \left| z_3 \frac{f_3''(z_3)}{f_3'(z_3)} \right| \right)$

for all $z = (z_1, z_2, z_3) \in D_p^3 \setminus \{0\}$, then $f \in K(D_p^3)$.

Now let us give two examples to illustrate the application of Theorem 1 in the following.

Example 1. Suppose that $p_j \geq p_1 \geq 2 (j = 2, \dots, n), 0 < |\lambda| \leq 1$, and k is a positive integer such that $k < \max\{p_j : j = 1, \dots, n\} \leq k + 1$, let

$$f(z) = (z_1 + a_1 z_1^2 + \sum_{j=2}^n a_j z_j^{k+1}, z_2 + a_2 z_2^2 + \sum_{j=3}^n a_j z_j^{k+1}, \frac{e^{\lambda z_3} - 1}{\lambda}, \dots, \frac{e^{\lambda z_n} - 1}{\lambda}),$$

where $a = \max\{|a_j| : j = 1, 2, \dots, n\}$. If $a \leq \frac{1-|\lambda|}{(k+1)^2+4} < \frac{1}{4}$ and

$$\left[\frac{p_1 + p_2}{1 - 2a} + \frac{p_1(k+1)a}{(1-2a)^2} \right] |a_j| \leq \frac{p_j(1-|\lambda|)}{(k+1)(k+|\lambda|)}, \quad j = 3, \dots, n,$$

then $f(z) \in K(D_p^n)$.

Proof Let

$$f_1(z) = z_1 + a_1 z_1^2 + \sum_{j=2}^n a_j z_j^{k+1}, \quad f_2(z) = z_2 + a_2 z_2^2 + \sum_{j=3}^n a_j z_j^{k+1}$$

and $f_j(z) = \frac{e^{\lambda z_j} - 1}{\lambda}$ for $j = 3, \dots, n$. Then it follows from $a = \max\{|a_j| : j = 1, 2, \dots, n\} < \frac{1}{4}$ that

$$\begin{aligned} \left| \frac{\partial f_1}{\partial z_1} \right| &= |1 + 2a_1 z_1| \geq 1 - 2a > 0, \quad \left| \frac{\partial f_2}{\partial z_2} \right| = |1 + 2a_2 z_2| \geq 1 - 2a > 0, \\ \left| \frac{z_1 \frac{\partial^2 f_1}{\partial z_1 \partial z_1}}{\frac{\partial f_1}{\partial z_1}} \right| &= \frac{|2a_1 z_1|}{|1 + 2a_1 z_1|} \leq \frac{2a}{1 - 2a} \leq 1, \quad \left| \frac{z_2 \frac{\partial^2 f_2}{\partial z_2 \partial z_2}}{\frac{\partial f_2}{\partial z_2}} \right| = \frac{|2a_2 z_2|}{|1 + 2a_2 z_2|} \leq \frac{2a}{1 - 2a} \leq 1, \\ \left| \frac{z_j f_j''(z_j)}{f_j'(z_j)} \right| &= |\lambda| |z_j| \leq 1, \quad j = 3, \dots, n. \end{aligned}$$

By calculating straightforwardly, we also obtain

$$\begin{aligned} & p_2 |z_2|^{p_2-2} \left(1 - \sum_{l=2}^n \left| \frac{z_2 \frac{\partial^2 f_2}{\partial z_2 \partial z_l}}{\frac{\partial f_2}{\partial z_2}} \right| \right) - p_1 \left(\sum_{l=1}^n \left| \frac{\frac{\partial^2 f_1}{\partial z_2 \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| + \sum_{l=2}^n \left| \frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_2}{\partial z_2}} \right| \left| \frac{\frac{\partial^2 f_2}{\partial z_2 \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| \right) \\ &= p_2 |z_2|^{p_2-2} \left(1 - \frac{|2a_2 z_2|}{|1 + 2a_2 z_2|} \right) - p_1 \left(\frac{|k(k+1)a_2 z_2^{k-1}|}{|1 + 2a_1 z_1|} + \frac{|(k+1)a_2 z_2^k|}{|1 + 2a_2 z_2|} \left| \frac{|2a_2|}{|1 + 2a_1 z_1|} \right| \right) \\ &\geq p_2 |z_2|^{p_2-2} \left(1 - \frac{2|a_2|}{1 - 2|a_2|} \right) - p_1 \left(\frac{k(k+1)|a_2|}{1 - 2|a_1|} |z_2|^{p_2-2} + \frac{(k+1)|a_2|}{1 - 2|a_2|} \frac{2|a_2|}{1 - 2|a_1|} |z_2|^{p_2-2} \right) \\ &\geq p_1 |z_2|^{p_2-2} \left(1 - \frac{2a}{1 - 2a} - \frac{k(k+1)a}{1 - 2a} - \frac{(k+1)a}{1 - 2a} \right) \\ &= \frac{p_1 |z_2|^{p_2-2}}{1 - 2a} \{1 - [(k+1)^2 + 4]a\} > 0. \end{aligned}$$

Since for $j = 3, \dots, n$, we have

$$\begin{aligned} \sum_{l=1}^n \left| \frac{\frac{\partial^2 f_1}{\partial z_j \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| &= \frac{|k(k+1)a_j z_j^{k-1}|}{|1 + 2a_1 z_1|} \leq \frac{k(k+1)|a_j|}{1 - 2|a_1|} |z_j|^{k-1} \leq \frac{k(k+1)|a_j|}{1 - 2|a_1|} |z_j|^{p_j-2}, \\ \sum_{l=2}^n \left| \frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_2}{\partial z_2}} \right| \left| \frac{\frac{\partial^2 f_2}{\partial z_j \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| &= \frac{|(k+1)a_2 z_2^k|}{|1 + 2a_2 z_2|} \frac{|k(k+1)a_j z_j^{k-1}|}{|1 + 2a_1 z_1|} \leq \frac{k(k+1)^2 |a_2| |a_j|}{(1 - 2|a_2|)(1 - 2|a_1|)} |z_j|^{p_j-2}, \\ \left| \frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_2}{\partial z_2}} \right| \left| \frac{\frac{\partial f_2}{\partial z_j}}{f_j''(z_j)} \right| &= \frac{|(k+1)a_2 z_2^k|}{|1 + 2a_1 z_1|} \frac{|(k+1)a_j z_j^k|}{|1 + 2a_2 z_2|} |\lambda| \leq \frac{(k+1)^2 |\lambda| |a_2| |a_j|}{(1 - 2|a_1|)(1 - 2|a_2|)} |z_j|^{p_j-2}, \\ \left| \frac{\frac{\partial f_1}{\partial z_j}}{\frac{\partial f_1}{\partial z_1}} \right| \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| &= \frac{|(k+1)a_j z_j^k|}{|1 + 2a_1 z_1|} |\lambda| \leq \frac{(k+1)|\lambda| |a_j|}{1 - 2|a_1|} |z_j|^{p_j-2}, \\ \sum_{l=2}^n \left| \frac{\frac{\partial^2 f_2}{\partial z_j \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| &= \frac{|k(k+1)a_j z_j^{k-1}|}{|1 + 2a_1 z_1|} \leq \frac{k(k+1)|a_j|}{1 - 2|a_1|} |z_j|^{p_j-2}, \\ \left| \frac{\frac{\partial f_2}{\partial z_j}}{\frac{\partial f_2}{\partial z_2}} \right| \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| &= \frac{|(k+1)a_j z_j^k|}{|1 + 2a_2 z_2|} |\lambda| \leq \frac{(k+1)|a_j| |\lambda|}{1 - 2|a_2|} |z_j|^{p_j-2}. \end{aligned}$$

Therefore

$$\begin{aligned}
& p_1 \left(\sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_j \partial z_l} \right| + \sum_{l=2}^n \left| \frac{\partial f_1}{\partial z_2} \right| \left| \frac{\partial^2 f_2}{\partial z_j \partial z_l} \right| + \left| \frac{\partial f_1}{\partial z_2} \right| \left| \frac{\partial f_2}{\partial z_2} \right| \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| + \left| \frac{\partial f_1}{\partial z_j} \right| \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| \right) \\
& + p_2 \left(\sum_{l=2}^n \left| \frac{\partial^2 f_2}{\partial z_j \partial z_l} \right| + \left| \frac{\partial f_2}{\partial z_j} \right| \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| \right) \\
& \leq p_1 |z_j|^{p_j-2} \left(\frac{k(k+1)|a_j|}{1-2|a_1|} + \frac{k(k+1)^2|a_2||a_j|}{(1-2|a_1|)(1-2|a_2|)} + \frac{(k+1)^2|\lambda||a_2||a_j|}{(1-2|a_1|)(1-2|a_2|)} \right. \\
& \quad \left. + \frac{(k+1)|a_j||\lambda|}{1-2|a_1|} \right) + p_2 |z_j|^{p_j-2} \left(\frac{k(k+1)|a_j|}{1-2|a_1|} + \frac{(k+1)|\lambda||a_j|}{1-2|a_2|} \right) \\
& \leq p_1 |z_j|^{p_j-2} \left(\frac{k(k+1)|a_j|}{1-2a} + \frac{k(k+1)^2 a |a_j|}{(1-2a)^2} + \frac{(k+1)^2 |\lambda| a |a_j|}{(1-2a)^2} + \frac{(k+1)|\lambda||a_j|}{1-2a} \right) \\
& \quad + p_2 |z_j|^{p_j-2} \left(\frac{k(k+1)|a_j|}{1-2a} + \frac{(k+1)|\lambda||a_j|}{1-2a} \right) \\
& = |z_j|^{p_j-2} (k+1)(k+|\lambda|) \left(\frac{p_1 |a_j|}{1-2a} + \frac{p_1 (k+1) a |a_j|}{(1-2a)^2} + \frac{p_2 |a_j|}{1-2a} \right) \\
& \leq |z_j|^{p_j-2} (k+1)(k+|\lambda|) \frac{p_j (1-|\lambda|)}{(k+1)(k+\lambda)} = p_j |z_j|^{p_j-2} (1-|\lambda|) \\
& \leq p_j |z_j|^{p_j-2} \left(1 - \left| \frac{z_j f_j''(z_j)}{f_j'(z_j)} \right| \right).
\end{aligned}$$

Hence it follows from Theorem 1 that $f \in K(D_p^n)$. \square

Example 2 Suppose that $p_j \geq p_1 \geq 2$ ($j = 2, \dots, n$), and k is a positive integer such that $k < \max\{p_j : j = 1, \dots, n\} \leq k+1$, let

$$f(z) = (z_1 + a_1 z_1^2 + \sum_{j=2}^n a_j z_j^{k+1}, z_2 + a_2 z_2^2 + \sum_{j=3}^n a_j z_j^2, z_3 + a_3 z_3^2, \dots, z_n + a_n z_n^2)$$

where $a = \max\{|a_j| : j = 1, 2, \dots, n\}$. If $a \leq \frac{1}{(k+1)^2+4} < \frac{1}{4}$, and

$$\left[\frac{p_1 + p_2}{1-2a} + \frac{p_1(k+1)a}{(1-2a)^2} \right] |a_j| \leq \frac{p_j(1 - \frac{2|a_j|}{1-2|a_j|})}{(k+1)(k + \frac{2|a_j|}{1-2|a_j|})}, \quad j = 3, \dots, n.$$

Then $f(z) \in K(D_p^n)$.

Next, we establish a sufficient condition for the biholomorphic convex mapping on D_p^n , which extend the main result of [6].

Theorem 2 Suppose that $n \geq 2, p_j \geq 2, j = 1, 2, \dots, n$. Let

$$f(z) = (p_1(z_1, z_2, \dots, z_n), p_2(z_2, z_n), \dots, p_{n-1}(z_{n-1}, z_n), p_n(z_n)),$$

where $z = (z_1, z_2, \dots, z_n) \in D_p^n, p_n(z_n) \in N(U), p_j(z_j, z_n) : D_p^2 \rightarrow C$ is holomorphic with $p_j(0, 0) = 0, \frac{\partial p_j}{\partial z_j}(0, 0) = 1, \frac{\partial p_j}{\partial z_n}(0, 0) = 0$, and $p_1(z_1, \dots, z_n) : D_p^n \rightarrow C$ is holomorphic with

$p_1(0, 0, \dots, 0) = 0$, $\frac{\partial p_1}{\partial z_1}(0, 0, \dots, 0) = 1$, $\frac{\partial p_1}{\partial z_l}(0, 0, \dots, 0) = 0$ for $2 \leq l \leq n$. If $f(z)$ satisfies the following conditions

$$\begin{aligned}
(1) \quad & \prod_{j=1}^{n-1} \frac{\partial p_j}{\partial z_j} p'_n(z_n) \neq 0, \quad |z_n p''_n(z_n)| \leq |p'_n(z_n)|; \\
(2) \quad & \sum_{l=1}^n \left| z_1 \frac{\partial^2 p_1}{\partial z_1 \partial z_l} \right| \leq \left| \frac{\partial p_1}{\partial z_1} \right|, \quad \left| z_j \frac{\partial^2 p_j}{\partial z_j^2} \right| + \left| z_j \frac{\partial^2 p_j}{\partial z_j \partial z_n} \right| \leq \left| \frac{\partial p_j}{\partial z_j} \right| (j = 2, 3, \dots, n-1); \\
(3) \quad & \frac{p_1}{\left| \frac{\partial p_1}{\partial z_1} \right|} \left(\frac{\left| \frac{\partial p_1}{\partial z_j} \right| \left(\left| \frac{\partial^2 p_j}{\partial z_j^2} \right| + \left| \frac{\partial^2 p_j}{\partial z_j \partial z_n} \right| \right)}{\left| \frac{\partial p_j}{\partial z_j} \right|} + \sum_{l=1}^n \left| \frac{\partial^2 p_1}{\partial z_j \partial z_l} \right| \right) \leq p_j |z_j|^{p_j-2} \left(1 - \frac{\left| z_j \frac{\partial^2 p_j}{\partial z_j^2} \right| + \left| z_j \frac{\partial^2 p_j}{\partial z_j \partial z_n} \right|}{\left| \frac{\partial p_j}{\partial z_j} \right|} \right) \\
& \quad (j = 2, 3, \dots, n-1); \\
(4) \quad & \sum_{j=2}^{n-1} \frac{p_j}{\left| \frac{\partial p_j}{\partial z_j} \right|} \left(\left| \frac{\partial^2 p_j}{\partial z_j \partial z_n} \right| + \left| \frac{\partial^2 p_j}{\partial z_n^2} \right| + \left| \frac{\partial p_j}{\partial z_n} \right| \left| \frac{p''_n(z_n)}{p'_n(z_n)} \right| \right) + \frac{p_1}{\left| \frac{\partial p_1}{\partial z_1} \right|} \left(\sum_{l=1}^n \left| \frac{\partial^2 p_1}{\partial z_n \partial z_l} \right| \right. \\
& \quad \left. + \sum_{j=2}^{n-1} \frac{\left| \frac{\partial p_1}{\partial z_j} \right| \left(\left| \frac{\partial^2 p_j}{\partial z_j \partial z_n} \right| + \left| \frac{\partial^2 p_j}{\partial z_n^2} \right| \right)}{\left| \frac{\partial p_j}{\partial z_j} \right|} + \left| \frac{\partial p_1}{\partial z_n} \right| \left| \frac{p''_n(z_n)}{p'_n(z_n)} \right| + \sum_{j=2}^{n-1} \left| \frac{\partial p_1}{\partial z_j} \right| \left| \frac{\partial p_j}{\partial z_n} \right| \left| \frac{p''_n(z_n)}{p'_n(z_n)} \right| \right) \\
& \quad \leq p_n |z_n|^{p_n-2} \left(1 - \left| \frac{z_n p''_n(z_n)}{p'_n(z_n)} \right| \right),
\end{aligned}$$

for all $z = (z_1, \dots, z_n) \in D_p^n \setminus \{0\}$, then $f \in K(D_p^n)$.

Proof By calculating the Fréchet derivatives of $f(z)$ straightforwardly, we obtain

$$Df(z) = \begin{pmatrix} \frac{\partial p_1}{\partial z_1} & \frac{\partial p_1}{\partial z_2} & \dots & \frac{\partial p_1}{\partial z_{n-1}} & \frac{\partial p_1}{\partial z_n} \\ 0 & \frac{\partial p_2}{\partial z_2} & \dots & 0 & \frac{\partial p_2}{\partial z_n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{\partial p_{n-1}}{\partial z_{n-1}} & \frac{\partial p_{n-1}}{\partial z_n} \\ 0 & 0 & \dots & 0 & p'_n(z_n) \end{pmatrix},$$

$$Df(z)^{-1} = \begin{pmatrix} \frac{1}{\frac{\partial p_1}{\partial z_1}} & -\frac{\frac{\partial p_1}{\partial z_2}}{\frac{\partial p_1}{\partial z_1} \frac{\partial p_2}{\partial z_2}} & \dots & -\frac{\frac{\partial p_1}{\partial z_{n-1}}}{\frac{\partial p_1}{\partial z_1} \frac{\partial p_{n-1}}{\partial z_{n-1}}} & -\frac{\frac{\partial p_1}{\partial z_n}}{\frac{\partial p_1}{\partial z_1} p'_n(z_n)} + \sum_{j=2}^{n-1} \frac{\frac{\partial p_j}{\partial z_n}}{\frac{\partial p_j}{\partial z_j} \frac{\partial p_1}{\partial z_1} p'_n(z_n)} \\ 0 & \frac{1}{\frac{\partial p_2}{\partial z_2}} & \dots & 0 & -\frac{\frac{\partial p_2}{\partial z_n}}{\frac{\partial p_2}{\partial z_2} p'_n(z_n)} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{\frac{\partial p_{n-1}}{\partial z_{n-1}}} & -\frac{\frac{\partial p_{n-1}}{\partial z_n}}{\frac{\partial p_{n-1}}{\partial z_{n-1}} p'_n(z_n)} \\ 0 & 0 & \dots & 0 & \frac{1}{p'_n(z_n)} \end{pmatrix},$$

$$\begin{aligned}
D^2 f(z)(b, b) &= \begin{pmatrix} \sum_{l=1}^n \frac{\partial^2 p_1}{\partial z_1 \partial z_l} b_l & \sum_{l=1}^n \frac{\partial^2 p_1}{\partial z_2 \partial z_l} b_l & \cdots & \sum_{l=1}^n \frac{\partial^2 p_1}{\partial z_{n-1} \partial z_l} b_l & C_1 \\ 0 & D_2 & \cdots & 0 & C_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & D_{n-1} & C_{n-1} \\ 0 & 0 & \cdots & 0 & C_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_{n-1} \\ b_n \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^n \sum_{l=1}^n \frac{\partial^2 p_1}{\partial z_j \partial z_l} b_l b_j \\ \frac{\partial^2 p_2}{\partial z_2^2} b_2^2 + 2 \frac{\partial^2 p_2}{\partial z_2 \partial z_n} b_2 b_n + \frac{\partial^2 p_2}{\partial z_n^2} b_n^2 \\ \cdots \\ \frac{\partial^2 p_{n-1}}{\partial z_{n-1}^2} b_{n-1}^2 + 2 \frac{\partial^2 p_{n-1}}{\partial z_{n-1} \partial z_n} b_{n-1} b_n + \frac{\partial^2 p_{n-1}}{\partial z_n^2} b_n^2 \\ p_n''(z_n) b_n^2 \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= \sum_{l=1}^n \frac{\partial^2 p_1}{\partial z_n \partial z_l} b_l, \\
C_j &= \frac{\partial^2 p_j}{\partial z_n \partial z_j} b_j + \frac{\partial^2 p_j}{\partial z_n^2} b_n \quad (j = 2, 3, \dots, n-1), \quad C_n = p_n''(z_n) b_n, \\
D_j &= \frac{\partial^2 p_j}{\partial z_j^2} b_j + \frac{\partial^2 p_j}{\partial z_j \partial z_n} b_n \quad (j = 2, 3, \dots, n-1).
\end{aligned}$$

Taking $z = (z_1, \dots, z_n) \in D_p^n \setminus \{0\}$, $b = (b_1, \dots, b_n) \in C^n$ such that

$$\operatorname{Re} \left\{ \sum_{j=1}^n p_j \left| \frac{z_j}{\rho(z)} \right|^{p_j} \frac{b_j}{z_j} \right\} = 0,$$

by the hypothesis of Theorem 2, we have

$$\begin{aligned}
J_f(z, b) &\geq \sum_{j=1}^n \frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - 2 \sum_{j=1}^n \frac{p_j}{\rho(z)} \left| \frac{z_j}{\rho(z)} \right|^{p_j} \operatorname{Re} \langle Df(z)^{-1} D^2 f(z)(b, b), \frac{\partial \rho}{\partial \bar{z}} \rangle \\
&= \sum_{j=1}^n \frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - \operatorname{Re} \left\{ \frac{1}{\frac{\partial p_1}{\partial z_1}} \left[\sum_{j=1}^n \sum_{l=1}^n \frac{\partial^2 p_1}{\partial z_j \partial z_l} b_l b_j - \sum_{j=2}^{n-1} \frac{\frac{\partial p_1}{\partial z_j}}{\frac{\partial p_j}{\partial z_j}} \left(\frac{\partial^2 p_j}{\partial z_j^2} b_j^2 + 2 \frac{\partial^2 p_j}{\partial z_j \partial z_n} b_j b_n \right. \right. \right. \\
&\quad \left. \left. + \frac{\partial^2 p_j}{\partial z_n^2} b_n^2 \right) - \frac{\partial p_1}{\partial z_n} \frac{p_n''(z_n)}{p_n'(z_n)} b_n^2 + \sum_{j=2}^{n-1} \frac{\frac{\partial p_j}{\partial z_n} \frac{\partial p_1}{\partial z_j}}{\frac{\partial p_j}{\partial z_j} \frac{\partial p_1}{\partial z_1}} \frac{p_n''(z_n)}{p_n'(z_n)} b_n^2 \right] \frac{p_1 |z_1|^{p_1}}{z_1 \rho(z)^{p_1}} \right. \\
&\quad \left. + \sum_{j=2}^{n-1} \frac{1}{\frac{\partial p_j}{\partial z_j}} \left(\frac{\partial^2 p_j}{\partial z_j^2} b_j^2 + 2 \frac{\partial^2 p_j}{\partial z_j \partial z_n} b_j b_n + \frac{\partial^2 p_j}{\partial z_n^2} b_n^2 - \frac{\partial p_j}{\partial z_n} \frac{p_n''(z_n)}{p_n'(z_n)} b_n^2 \right) \frac{p_j |z_j|^{p_j}}{z_j \rho(z)^{p_j}} + \frac{p_n''(z_n)}{p_n'(z_n)} b_n^2 \frac{p_n |z_n|^{p_n}}{z_n \rho(z)^{p_n}} \right\} \\
&\geq \sum_{j=1}^n \frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - \frac{1}{\left| \frac{\partial p_1}{\partial z_1} \right|} \left[\sum_{j=1}^n \sum_{l=1}^n \left| \frac{\partial^2 p_1}{\partial z_j \partial z_l} \right| |b_j|^2 + \sum_{j=2}^{n-1} \left| \frac{\frac{\partial p_1}{\partial z_j}}{\frac{\partial p_j}{\partial z_j}} \right| \left(\left| \frac{\partial^2 p_j}{\partial z_j^2} \right| |b_j|^2 + \left| \frac{\partial^2 p_j}{\partial z_j \partial z_n} \right| |b_j|^2 \right. \right. \\
&\quad \left. \left. + \left| \frac{\partial^2 p_j}{\partial z_j \partial z_n} \right| |b_n|^2 + \left| \frac{\partial^2 p_j}{\partial z_n^2} \right| |b_n|^2 \right) + \left| \frac{\partial p_1}{\partial z_n} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| |b_n|^2 + \sum_{j=2}^{n-1} \left| \frac{\frac{\partial p_j}{\partial z_n} \frac{\partial p_1}{\partial z_j}}{\frac{\partial p_j}{\partial z_j} \frac{\partial p_1}{\partial z_1}} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| |b_n|^2 \right] \frac{p_1 |z_1|^{p_1-1}}{\rho(z)^{p_1}}
\end{aligned}$$

$$\begin{aligned}
& - \sum_{j=2}^{n-1} \frac{1}{|\frac{\partial p_j}{\partial z_j}|} \left(\left| \frac{\partial^2 p_j}{\partial z_j^2} \right| |b_j|^2 + \left| \frac{\partial^2 p_j}{\partial z_j \partial z_n} \right| |b_j|^2 + \left| \frac{\partial^2 p_j}{\partial z_j \partial z_n} \right| |b_n|^2 \right. \\
& \quad \left. + \left| \frac{\partial^2 p_j}{\partial z_n^2} \right| |b_n|^2 + \left| \frac{\partial p_j}{\partial z_n} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| |b_n|^2 \right) \frac{p_j |z_j|^{p_j-1}}{\rho(z)^{p_j}} - \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| |b_n|^2 \frac{p_n |z_n|^{p_n-1}}{\rho(z)^{p_n}} \\
& = |b_1|^2 \frac{p_1 |z_1|^{p_1-2}}{\rho(z)^{p_1}} \left(1 - \frac{\sum_{l=1}^n |z_1 \frac{\partial^2 p_1}{\partial z_1 \partial z_l}|}{|\frac{\partial p_1}{\partial z_1}|} \right) + \sum_{j=2}^{n-1} |b_j|^2 \left[\frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} \left(1 - \frac{|z_j \frac{\partial^2 p_j}{\partial z_j^2}| + |z_j \frac{\partial^2 p_j}{\partial z_j \partial z_n}|}{|\frac{\partial p_j}{\partial z_j}|} \right) \right. \\
& \quad \left. - \frac{p_1 |z_1|^{p_1-1}}{\rho(z)^{p_1} |\frac{\partial p_1}{\partial z_1}|} \left(\sum_{l=1}^n \left| \frac{\partial^2 p_1}{\partial z_j \partial z_l} \right| + \left| \frac{\partial p_1}{\partial z_j} \right| \frac{|\frac{\partial^2 p_j}{\partial z_j^2}| + |\frac{\partial^2 p_j}{\partial z_j \partial z_n}|}{|\frac{\partial p_j}{\partial z_j}|} \right) \right] + |b_n|^2 \left[\frac{p_n |z_n|^{p_n-2}}{\rho(z)^{p_n}} \left(1 - \left| \frac{z_n p_n''(z_n)}{p_n'(z_n)} \right| \right) \right. \\
& \quad \left. - \frac{p_1 |z_1|^{p_1-1}}{\rho(z)^{p_1} |\frac{\partial p_1}{\partial z_1}|} \left(\sum_{l=1}^n \left| \frac{\partial^2 p_1}{\partial z_n \partial z_l} \right| + \sum_{j=2}^{n-1} \left| \frac{\partial p_1}{\partial z_j} \right| \frac{|\frac{\partial^2 p_j}{\partial z_n^2}| + |\frac{\partial^2 p_j}{\partial z_j \partial z_n}|}{|\frac{\partial p_j}{\partial z_j}|} + \left| \frac{\partial p_1}{\partial z_n} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| + \sum_{j=2}^{n-1} \left| \frac{\partial p_1}{\partial z_j} \right| \left| \frac{\frac{\partial p_j}{\partial z_n}}{\frac{\partial p_j}{\partial z_j}} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| \right) \right. \\
& \quad \left. - \sum_{j=2}^{n-1} \left| \frac{\frac{\partial^2 p_j}{\partial z_j \partial z_n}}{\frac{\partial p_j}{\partial z_j}} \right| \frac{p_j |z_j|^{p_j-1}}{\rho(z)^{p_j}} - \sum_{j=2}^{n-1} \left| \frac{\frac{\partial^2 p_j}{\partial z_n^2}}{\frac{\partial p_j}{\partial z_j}} \right| \frac{p_j |z_j|^{p_j-1}}{\rho(z)^{p_j}} - \sum_{j=2}^{n-1} \left| \frac{\frac{\partial p_j}{\partial z_n}}{\frac{\partial p_j}{\partial z_j}} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| \frac{p_j |z_j|^{p_j-1}}{\rho(z)^{p_j}} \right] \\
& \geq |b_1|^2 \frac{p_1 |z_1|^{p_1-2}}{\rho(z)^{p_1}} \left(1 - \frac{\sum_{l=1}^n |z_1 \frac{\partial^2 p_1}{\partial z_1 \partial z_l}|}{|\frac{\partial p_1}{\partial z_1}|} \right) + \sum_{j=2}^{n-1} |b_j|^2 \left[p_j |z_j|^{p_j-2} \left(1 - \frac{|z_j \frac{\partial^2 p_j}{\partial z_j^2}| + |z_j \frac{\partial^2 p_j}{\partial z_j \partial z_n}|}{|\frac{\partial p_j}{\partial z_j}|} \right) \right. \\
& \quad \left. - \frac{p_1}{|\frac{\partial p_1}{\partial z_1}|} \left(\sum_{l=1}^n \left| \frac{\partial^2 p_1}{\partial z_j \partial z_l} \right| + \left| \frac{\partial p_1}{\partial z_j} \right| \frac{|\frac{\partial^2 p_j}{\partial z_j^2}| + |\frac{\partial^2 p_j}{\partial z_j \partial z_n}|}{|\frac{\partial p_j}{\partial z_j}|} \right) \right] + |b_n|^2 \left[p_n |z_n|^{p_n-2} \left(1 - \left| \frac{z_n p_n''(z_n)}{p_n'(z_n)} \right| \right) \right. \\
& \quad \left. - \frac{p_1}{|\frac{\partial p_1}{\partial z_1}|} \left(\sum_{l=1}^n \left| \frac{\partial^2 p_1}{\partial z_n \partial z_l} \right| + \sum_{j=2}^{n-1} \left| \frac{\partial p_1}{\partial z_j} \right| \frac{|\frac{\partial^2 p_j}{\partial z_n^2}| + |\frac{\partial^2 p_j}{\partial z_j \partial z_n}|}{|\frac{\partial p_j}{\partial z_j}|} + \left| \frac{\partial p_1}{\partial z_n} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| + \sum_{j=2}^{n-1} \left| \frac{\partial p_1}{\partial z_j} \right| \left| \frac{\frac{\partial p_j}{\partial z_n}}{\frac{\partial p_j}{\partial z_j}} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| \right) \right. \\
& \quad \left. - \sum_{j=2}^{n-1} \left| \frac{\frac{\partial^2 p_j}{\partial z_j \partial z_n}}{\frac{\partial p_j}{\partial z_j}} \right| p_j - \sum_{j=2}^{n-1} \left| \frac{\frac{\partial^2 p_j}{\partial z_n^2}}{\frac{\partial p_j}{\partial z_j}} \right| p_j - \sum_{j=2}^{n-1} \left| \frac{\frac{\partial p_j}{\partial z_n}}{\frac{\partial p_j}{\partial z_j}} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| p_j \right] \\
& \geq 0.
\end{aligned}$$

Thus it follows from Theorem A that $f \in K(D_p^n)$. The proof is complete. \square

Remark 2 Setting $p_j(z_j, z_n) = f_j(z_j) + p_j(z_n)$ $j = 2, \dots, n$ in Theorem 2, we get Theorem 2.1 of [6]. Setting $n = 3$ in Theorem 2, we get Corollary 1.

Example 3 Suppose that $p_j \geq p_1 \geq 2$ ($j = 2, \dots, n$), $0 < |\lambda| \leq 1$, and k is a positive integer such that $k < \max\{p_j : j = 1, \dots, n\} \leq k+1$. Let

$$p(z) = (z_1 + \sum_{j=2}^{n-1} a_j z_j^{k+1} + a_n z_1 z_n^{k+1}, z_2 + a_2 z_2 z_n^{k+1}, \dots, z_{n-1} + a_{n-1} z_{n-1} z_n^{k+1}, \frac{e^{\lambda z_n - 1}}{\lambda}),$$

where $a = \max \left\{ |a_j| : j = 2, \dots, n \right\}$. If $a \leq \frac{1-|\lambda|}{2(k+1)^2(k+1+|\lambda|)+1+|\lambda|} < 1$, and

$$\sum_{j=2}^{n-1} \frac{p_j |a_j|}{1-|a_j|} + \frac{a p_1}{1-a} \left(1 + (k+1) \sum_{j=2}^{n-1} \frac{|a_j|}{1-|a_j|} \right) \leq \frac{p_n(1-|\lambda|)}{(k+1)(k+1+|\lambda|)},$$

then $p(z) \in K(D_p^n)$.

Proof Put

$$p_1(z_1, z_2, \dots, z_n) = z_1 + \sum_{j=2}^{n-1} a_j z_j^{k+1} + a_n z_1 z_n^{k+1},$$

$$p_j(z_j, z_n) = z_j + a_j z_j z_n^{k+1} \quad (j = 2, 3, \dots, n-1), \quad p_n(z_n) = \frac{e^{\lambda z_n} - 1}{\lambda}.$$

Then it follows from $a = \max \left\{ |a_j| : j = 2, \dots, n \right\} \leq \frac{1-|\lambda|}{2(k+1)^2(k+1+|\lambda|)+1-|\lambda|} < \frac{1}{k+2} < 1$ that

$$\begin{aligned} \left| \frac{\partial p_1}{\partial z_1} \right| &= |1 + a_n z_n^{k+1}| \geq 1 - |a_n| \geq 1 - a > 0, \\ \left| \frac{\partial p_j}{\partial z_j} \right| &= |1 + a_j z_n^{k+1}| \geq 1 - |a_j| \geq 1 - a > 0, \\ p'_n(z_n) &= e^{\lambda z_n} \neq 0, \\ \left| \frac{\partial p_1}{\partial z_1} \right| - \sum_{l=1}^n \left| z_1 \frac{\partial^2 p_1}{\partial z_1 \partial z_l} \right| &= |1 + a_n z_n^{k+1}| - |(k+1) a_n z_1 z_n^k| \geq 1 - (k+2) |a_n| > 0, \\ \left| \frac{\partial p_j}{\partial z_j} \right| - \left| z_j \frac{\partial^2 p_j}{\partial z_j^2} \right| - \left| \frac{z_j \partial^2 p_j}{\partial z_j \partial z_n} \right| &= |1 + a_j z_n^{k+1}| - (k+1) |a_j z_j z_n^k| \geq 1 - (k+2) |a_j| > 0, \end{aligned}$$

and $\left| \frac{z_n p''_n(z_n)}{p'_n(z_n)} \right| = |\lambda| |z| \leq |\lambda| < 1$. By calculating straightforwardly, we also obtain

$$\begin{aligned} & p_j |z_j|^{p_j-2} \left(1 - \frac{\left| z_j \frac{\partial^2 p_j}{\partial z_j^2} \right| + \left| z_j \frac{\partial^2 p_j}{\partial z_j \partial z_n} \right|}{\left| \frac{\partial p_j}{\partial z_j} \right|} \right) - \frac{p_1}{\left| \frac{\partial p_1}{\partial z_1} \right|} \left(\frac{\left| \frac{\partial p_1}{\partial z_j} \right| \left(\left| \frac{\partial^2 p_j}{\partial z_j^2} \right| + \left| \frac{\partial^2 p_j}{\partial z_j \partial z_n} \right| \right)}{\left| \frac{\partial p_j}{\partial z_j} \right|} + \sum_{l=1}^n \left| \frac{\partial^2 p_1}{\partial z_j \partial z_l} \right| \right) \\ &= p_j |z_j|^{p_j-2} \left(1 - \frac{|(k+1) z_j a_j z_n^k|}{|1 + a_j z_n^{k+1}|} \right) - \frac{p_1}{|1 + a_n z_n^{k+1}|} \left(\frac{|(k+1) a_j z_j^k| |(k+1) a_j z_n^k|}{|1 + a_j z_n^{k+1}|} + |k(k+1) a_j z_j^{k-1}| \right) \\ &\geq p_j |z_j|^{p_j-2} \left(1 - \frac{(k+1) |a_j|}{1 - |a_j|} \right) - \frac{p_j}{1 - |a_n|} \left(\frac{(k+1)^2 |a_j|^2}{1 - |a_j|} + k(k+1) |a_j| \right) |z_j|^{k-1} \\ &\geq p_j |z_j|^{p_j-2} \left(1 - \frac{(k+1)a}{1-a} \right) - \frac{p_j}{1-a} \left(\frac{(k+1)^2 a^2}{1-a} + (k+1)^2 a \right) |z_j|^{p_j-2} \\ &= \frac{p_j |z_j|^{p_j-2}}{1-a} \left(1 - (k+2)a - \frac{(k+1)^2 a}{1-a} \right) \\ &\geq \frac{p_j |z_j|^{p_j-2}}{1-a} \left(1 - \frac{(k+2)(1-|\lambda|)}{2(k+1)^2(k+1+|\lambda|)+1-|\lambda|} - \frac{1-|\lambda|}{2(k+1+|\lambda|)} \right) \end{aligned}$$

$$\geq \frac{p_j |z_j|^{p_j-2}}{1-a} \left[1 - \frac{[(k+2) + (k+1)^2](1-|\lambda|)}{2(k+1)^2(k+1+|\lambda|)} \right] > 0, \quad j = 2, \dots, n-1.$$

Since

$$\begin{aligned} \sum_{j=2}^{n-1} p_j \left| \frac{\frac{\partial^2 p_j}{\partial z_j \partial z_n}}{\frac{\partial p_j}{\partial z_j}} \right| &= \sum_{j=2}^{n-1} p_j \left| \frac{(k+1)a_j z_n^k}{1+a_j z_n^{k+1}} \right| \leq (k+1) \sum_{j=2}^{n-1} \frac{p_j |a_j|}{1-|a_j|} |z_n|^{p_n-2}, \\ \sum_{j=2}^{n-1} p_j \left| \frac{\frac{\partial^2 p_j}{\partial z_n^2}}{\frac{\partial p_j}{\partial z_j}} \right| &= \sum_{j=2}^{n-1} p_j \left| \frac{k(k+1)a_j z_j z_n^{k-1}}{1+a_j z_n^{k+1}} \right| \leq k(k+1) \sum_{j=2}^{n-1} \frac{p_j |a_j|}{1-|a_j|} |z_n|^{p_n-2}, \\ \sum_{j=2}^{n-1} p_j \left| \frac{\frac{\partial p_j}{\partial z_n}}{\frac{\partial p_j}{\partial z_j}} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| &= \sum_{j=2}^{n-1} p_j \left| \frac{(k+1)a_j z_j z_n^k}{1+a_j z_n^{k+1}} \right| |\lambda| \leq |\lambda|(k+1) \sum_{j=2}^{n-1} \frac{p_j |a_j|}{1-|a_j|} |z_n|^{p_n-2}, \end{aligned}$$

we have

$$\begin{aligned} & \sum_{j=2}^{n-1} \frac{p_j}{\left| \frac{\partial p_j}{\partial z_j} \right|} \left(\left| \frac{\partial^2 p_j}{\partial z_j \partial z_n} \right| + \left| \frac{\partial^2 p_j}{\partial z_n^2} \right| + \left| \frac{\partial p_j}{\partial z_n} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| \right) + \frac{p_1}{\left| \frac{\partial p_1}{\partial z_1} \right|} \left(\sum_{l=1}^n \left| \frac{\partial^2 p_1}{\partial z_n \partial z_l} \right| \right. \\ & \quad \left. + \sum_{j=2}^{n-1} \frac{\left| \frac{\partial p_1}{\partial z_j} \right| \left(\left| \frac{\partial^2 p_j}{\partial z_j \partial z_n} \right| + \left| \frac{\partial^2 p_j}{\partial z_n^2} \right| \right)}{\left| \frac{\partial p_j}{\partial z_j} \right|} + \left| \frac{\partial p_1}{\partial z_n} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| + \sum_{j=2}^{n-1} \left| \frac{\partial p_1}{\partial z_j} \right| \left| \frac{\frac{\partial p_j}{\partial z_n}}{\frac{\partial p_j}{\partial z_j}} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| \right) \\ & \leq |z_n|^{p_n-2} \left[(k+1) \sum_{j=2}^{n-1} \frac{p_j |a_j|}{1-|a_j|} + k(k+1) \sum_{j=2}^{n-1} \frac{p_j |a_j|}{1-|a_j|} + |\lambda|(k+1) \sum_{j=2}^{n-1} \frac{p_j |a_j|}{1-|a_j|} \right] \\ & \quad + \frac{p_1}{1-|a_n|} \left[(k+1)|a_n| + k(k+1)|a_n| + \sum_{j=2}^{n-1} \frac{(k+1)|a_j|[(k+1)|a_j| + k(k+1)|a_j|]}{1-|a_j|} \right. \\ & \quad \left. + (k+1)|a_n||\lambda| + \sum_{j=2}^{n-1} (k+1)|a_j| \frac{(k+1)|a_j||\lambda|}{1-|a_j|} \right] |z_n|^{k-1} \\ & \leq |z_n|^{p_n-2} (k+1)(k+1+|\lambda|) \sum_{j=2}^{n-1} \frac{p_j |a_j|}{1-|a_j|} + \frac{p_1}{1-a} \left[(k+1)a + k(k+1)a \right. \\ & \quad \left. + \sum_{j=2}^{n-1} \frac{(k+1)|a_j|(k+1)^2|a_j|}{1-|a_j|} + (k+1)a|\lambda| + \sum_{j=2}^{n-1} \frac{(k+1)|a_j|(k+1)|a_j||\lambda|}{1-|a_j|} \right] |z_n|^{p_n-2} \\ & \leq |z_n|^{p_n-2} (k+1)(k+1+|\lambda|) \sum_{j=2}^{n-1} \frac{p_j |a_j|}{1-|a_j|} \\ & \quad + \frac{p_1}{1-a} (k+1)(k+1+|\lambda|)a |z_n|^{p_n-2} \left[1 + (k+1) \sum_{j=2}^{n-1} \frac{|a_j|}{1-|a_j|} \right] \\ & \leq |z_n|^{p_n-2} (k+1)(k+1+|\lambda|) \left[\sum_{j=2}^{n-1} \frac{p_j |a_j|}{1-|a_j|} + \frac{a p_1}{1-a} \left(1 + (k+1) \sum_{j=2}^{n-1} \frac{|a_j|}{1-|a_j|} \right) \right] \\ & \leq p_n |z_n|^{p_n-2} (1-|\lambda|) \leq p_n |z_n|^{p_n-2} \left(1 - \left| \frac{z_n p_n''(z_n)}{p_n'(z_n)} \right| \right), \end{aligned}$$

hence by Theorem 2, we obtain that $f \in K(D_p^n)$. \square

By applying the same method of the proof for example 3, we only need to let $\frac{2|a_n|}{1-2|a_n|}$ instead of $|\lambda|$, we may prove the following result.

Example 4 Suppose that $p_j \geq p_1 \geq 2 (j = 2, \dots, n)$, $0 < |a_n| \leq \frac{1}{4}$, and k is a positive integer such that $k < \max\{p_j : j = 1, \dots, n\} \leq k + 1$. Let

$$p(z) = (z_1 + \sum_{j=2}^{n-1} a_j z_j^{k+1} + a_n z_1 z_n^{k+1}, z_2 + a_2 z_n z_n^{k+1}, \dots, z_{n-1} + a_{n-1} z_{n-1} z_n^{k+1}, z_n + a_n z_n^2),$$

where $a = \max \left\{ |a_j| : j = 2, \dots, n \right\}$. If

$$a \leq \frac{1 - \frac{2|a_n|}{1-2|a_n|}}{2(k+1)^2(k+1 + \frac{2|a_n|}{1-2|a_n|}) + 1 - \frac{2|a_n|}{1-2|a_n|}} < 1,$$

and

$$\sum_{j=2}^{n-1} \frac{p_j |a_j|}{1 - |a_j|} + \frac{p_1 a}{1 - a} \left[1 + (k+1) \sum_{j=2}^{n-1} \frac{|a_j|}{1 - |a_j|} \right] \leq \frac{p_n (1 - \frac{2|a_n|}{1-2|a_n|})}{(k+1)(k+1 + \frac{2|a_n|}{1-2|a_n|})},$$

then $p(z) \in K(D_p^n)$.

Applying the same method as our proof of Theorem 1, we may show the following theorem.

Theorem 3 Suppose that $n \geq 2, p_j \geq 2, j = 1, 2, \dots, n$, k is a positive integer such that $2 \leq k \leq n$, and $f_j, p_j : U \rightarrow C$ is holomorphic with $f_j(0) = 0, f'_j(0) = 0 (j = 2, 3, \dots, n), f_k(z_k) = 0, p_1(z_1, z_2, \dots, z_n) : D_p \rightarrow C$ is holomorphic with $p_1(0, 0, \dots, 0) = 0, \frac{\partial p_1}{\partial z_1}(0, 0, \dots, 0) = 1, \frac{\partial p_l}{\partial z_l}(0, 0, \dots, 0) = 0 (l = 2, 3, \dots, n)$. Let

$$\begin{aligned} f(z) &= (p_1(z_1, z_2, \dots, z_n), p_2(z_2) + f_2(z_k), \dots, p_{k-1}(z_{k-1}) + f_{k-1}(z_k), \\ &\quad p_k(z_k), p_{k+1}(z_{k+1}) + f_{k+1}(z_k), \dots, p_n(z_n) + f_n(z_k)). \end{aligned}$$

If for any $z = (z_1, z_2, \dots, z_n) \in D_p^n \setminus \{0\}$, we have

$$(1) \quad \frac{\partial p_1}{\partial z_1} \cdot \prod_{j=2}^n p'_j(z_j) \neq 0, \quad |z_j p''_j(z_j)| \leq |p'_j(z_j)|, j = 2, \dots, n;$$

$$(2) \quad \sum_{l=1}^n \left| z_1 \frac{\partial^2 p_1}{\partial z_1 \partial z_l} \right| \leq \left| \frac{\partial p_1}{\partial z_1} \right|;$$

$$(3) \quad p_1 \left| \frac{\frac{\partial p_1}{\partial z_j} \cdot \frac{p''_j(z_j)}{p'_j(z_j)}}{\frac{\partial p_1}{\partial z_1}} \right| + p_1 \sum_{l=1}^n \left| \frac{\frac{\partial^2 p_1}{\partial z_j \partial z_l}}{\frac{\partial p_1}{\partial z_1}} \right| \leq p_j |z_j|^{p_j-2} \left(1 - \left| \frac{z_j p''_j(z_j)}{p'_j(z_j)} \right| \right),$$

($j = 2, \dots, k-1, k+1, \dots, n-1$);

$$(4) \quad \sum_{j=2, j \neq k}^n \left| \frac{f''_j(z_k)}{p'_j(z_j)} \right| p_j + \sum_{j=2, j \neq k}^n \left| \frac{f'_j(z_k)}{p'_j(z_j)} \right| \left| \frac{p''_k(z_k)}{p'_k(z_k)} \right| p_j + \sum_{j=2, j \neq k}^n \left| \frac{\frac{f''_j(z_k)}{p'_j(z_j)} \frac{\partial p_1}{\partial z_j}}{\frac{\partial p_1}{\partial z_1}} \right| p_1$$

$$\begin{aligned}
& + \sum_{j=2, j \neq k}^n \left| \frac{\frac{f'_j(z_k)}{p'_j(z_j)} \frac{p''_k(z_k)}{p'_k(z_k)} \frac{\partial p_1}{\partial z_j}}{\frac{\partial p_1}{\partial z_1}} \right| p_1 + \sum_{l=1}^n \left| \frac{\frac{\partial^2 p_1}{\partial z_l \partial z_k}}{\frac{\partial p_1}{\partial z_1}} \right| p_1 + \left| \frac{\frac{p''_k(z_k)}{p'_k(z_k)} \frac{\partial p_1}{\partial z_k}}{\frac{\partial p_1}{\partial z_1}} \right| p_1 \\
& \leq \left(1 - \left| \frac{z_k p''_k(z_k)}{p'_k(z_k)} \right| \right) p_k |z_k|^{p_k-2},
\end{aligned}$$

then $f \in K(D_p^n)$.

Finally, we give a partial answer to Problem II by verifying the following theorem.

Theorem 4 Suppose that $p \geq 2$. Let

$$f(z) = (z_1 + a_1 z_1^2 + a'_1 z_2^2, a_2 z_1^2 + z_2 + a'_2 z_2^2),$$

where $z = (z_1, z_2)$. If (a_1, a_2, a'_1, a'_2) satisfies the following conditions:

$$4|a_1| + 2|a'_2| + 8|a_1||a'_2| + 2|a_2| + 4|a_1||a_2| + 8|a'_1||a_2| + 4|a_1||a_2| \leq 1, \quad (4)$$

$$2|a_1| + 2|a'_1| + 4|a'_1||a'_2| + 4|a'_2| + 8|a_1||a'_2| + 4|a'_1||a'_2| + 8|a'_1||a_2| \leq 1. \quad (5)$$

Then $f \in K(B_p^2)$.

Proof By calculating the Fréchet derivatives of $f(z)$ straightforwardly, we obtain

$$\begin{aligned}
Df(z) &= \begin{pmatrix} 1 + 2a_1 z_1 & 2a'_1 z_2 \\ 2a_2 z_1 & 1 + 2a'_2 z_2 \end{pmatrix}, \quad Df(z)^{-1} = \begin{pmatrix} \frac{1+2a'_2 z_2}{A} & \frac{-2a'_1 z_2}{A} \\ \frac{-2a_2 z_1}{A} & \frac{1+2a_1 z_1}{A} \end{pmatrix}, \\
D^2 f(z)(b, b) &= \begin{pmatrix} 2a_1 b_1 & 2a'_1 b_2 \\ 2a_2 b_1 & 2a'_2 b_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 2a_1 b_1^2 + 2a'_1 b_2^2 \\ 2a_2 b_1^2 + 2a'_2 b_2^2 \end{pmatrix}, \\
Df(z)^{-1} D^2 f(z)(b, b) &= \begin{pmatrix} \frac{(1+2a'_2 z_2)(2a_1 b_1^2 + 2a'_1 b_2^2)}{A} - \frac{2a'_1 z_2(2a_2 b_1^2 + 2a'_2 b_2^2)}{A} \\ -\frac{2a_2 z_1(2a_1 b_1^2 + 2a'_1 b_2^2)}{A} + \frac{(1+2a_1 z_1)(2a_2 b_1^2 + 2a'_2 b_2^2)}{A} \end{pmatrix},
\end{aligned}$$

where

$$A = 4a_1 a'_2 z_1 z_2 - 4a'_1 a_2 z_1 z_2 + 1 + 2a'_2 z_2 + 2a_1 z_1. \quad (6)$$

Taking $z = (z_1, z_2) \in B_p^2 \setminus \{0\}$, $b = (b_1, b_2) \in C^2$ such that $\operatorname{Re}\langle b, \frac{\partial u}{\partial \bar{z}} \rangle = 0$, by the hypothesis of Theorem 2, we have

$$\begin{aligned}
\frac{2}{p} J_f(z, b) &\geq \sum_{j=1}^2 |b_j|^2 |z_j|^{p-2} - \frac{2}{p} \operatorname{Re}\langle Df(z)^{-1} D^2 f(z)(b, b), \frac{\partial u}{\partial \bar{z}} \rangle \\
&= \sum_{j=1}^2 |b_j|^2 |z_j|^{p-2} - \operatorname{Re}\left\{ \left(\frac{(1+2a'_2 z_2)(2a_1 b_1^2 + 2a'_1 b_2^2) - 2a'_1 z_2(2a_2 b_1^2 + 2a'_2 b_2^2)}{A} \right) \frac{|z_1|^p}{z_1} \right. \\
&\quad \left. + \left(\frac{(1+2a_1 z_1)(2a_2 b_1^2 + 2a'_2 b_2^2) - 2a_2 z_1(2a_1 b_1^2 + 2a'_1 b_2^2)}{A} \right) \frac{|z_2|^p}{z_2} \right\} \\
&\geq \sum_{j=1}^2 |b_j|^2 |z_j|^{p-2} - \left\{ \left(\left| \frac{2a_1 + 4a_1 a'_2 z_2}{A} \right| |b_1|^2 + \left| \frac{2a'_1 + 4a'_1 a'_2 z_2}{A} \right| |b_2|^2 + \left| \frac{4a'_1 a_2 z_2}{A} \right| |b_1|^2 \right. \right. \\
&\quad \left. \left. + \left| \frac{4a'_1 a'_2}{A} \right| |b_2|^2 \right) |z_1|^{p-1} + \left(\left| \frac{4a_1 a_2 z_1}{A} \right| |b_1|^2 + \left| \frac{4a'_1 a_2 z_1}{A} \right| |b_2|^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left| \frac{2a_2 + 4a_1a_2z_1}{A} \right| |b_1|^2 + \left| \frac{2a'_2 + 4a_1a'_2z_1}{A} \right| |b_2|^2 \Big) |z_2|^{p-1} \Big\} \\
\geq & |b_1|^2 |z_1|^{p-2} \left(1 - \frac{2|a_1| + 4|a_1||a'_2| + 2|a_2| + 4|a_1||a_2| + 4|a'_1||a_2| + 4|a_1||a_2|}{1 - 4|a_1||a'_2| - 4|a'_1||a_2| - 2|a'_2| - 2|a_1|} \right) \\
& + |b_2|^2 |z_2|^{p-2} \left(1 - \frac{2|a'_1| + 4|a'_1||a'_2| + 2|a'_2| + 4|a_1||a'_2| + 4|a'_1||a'_2| + 4|a'_1||a_2|}{1 - 4|a_1||a'_2| - 4|a'_1||a_2| - 2|a'_2| - 2|a_1|} \right) \\
\geq & 0,
\end{aligned}$$

hence it follows from Theorem A that $f(z) \in K(B_p^2)$. \square

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